
CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2017 Trial Examination

FORM VI

MATHEMATICS EXTENSION 1

Friday 4th August 2017

General Instructions

- Reading time — 5 minutes
- Writing time — 2 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 70 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 60 Marks

- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Reference sheet
- Candidature — 125 boys

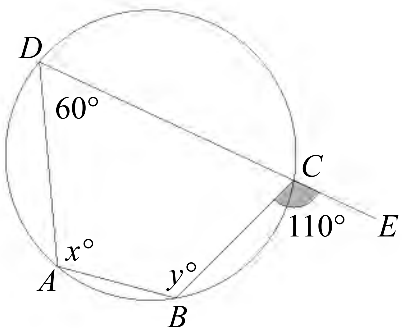
Examiner

FMW

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE



Suppose $ABCD$ is a cyclic quadrilateral with DC produced to E. What are the values of x and y?

(A) $x = 120$, $y = 110$

(B) $x = 110$, $y = 110$

(C) $x = 120$, $y = 120$

(D) $x = 110$, $y = 120$

QUESTION TWO

Let $A = (-3, 2)$ and $B = (4, -7)$. The interval AB is divided externally in the ratio $5 : 3$ by the point $P(x, y)$. What is the value of x?

(A) $14\frac{1}{2}$

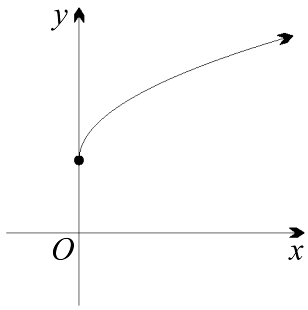
(B) 13

(C) $1\frac{3}{8}$

(D) $-13\frac{1}{2}$

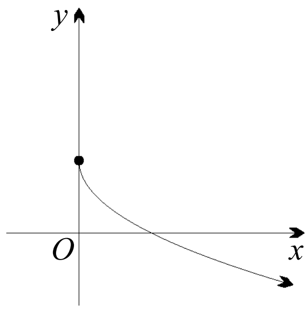
Examination continues next page ...

QUESTION THREE

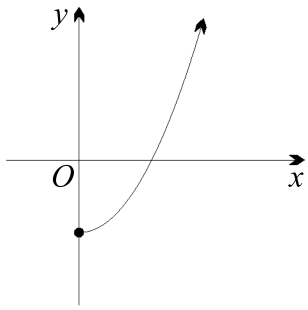


The diagram shows the graph of $y = f(x)$. Which diagram shows the graph of $y = f^{-1}(x)$?

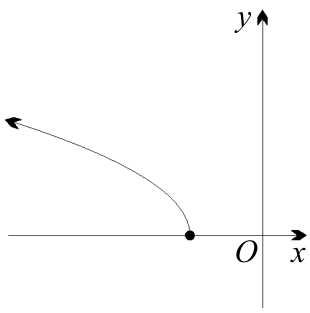
(A)



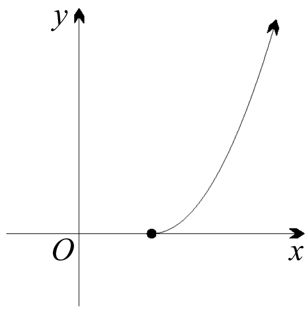
(B)



(C)



(D)



Examination continues overleaf ...

QUESTION FOUR

What is the derivative of $\sin^{-1} 3x$?

(A) $\dfrac{1}{3\sqrt{1-9x^2}}$

(B) $\dfrac{-1}{3\sqrt{1-3x^2}}$

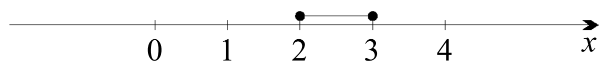
(C) $\dfrac{3}{\sqrt{1-9x^2}}$

(D) $\dfrac{3}{\sqrt{1-3x^2}}$

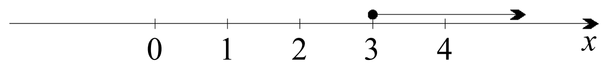
QUESTION FIVE

Which number line graph shows the correct solution to $\dfrac{x}{x-2} \geq 3$?

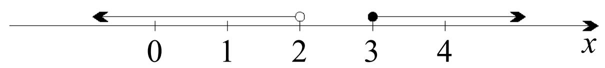
(A)



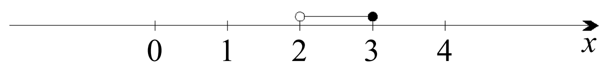
(B)



(C)



(D)



Examination continues next page ...

QUESTION SIX

What is the domain of the function $y = 4\sin^{-1}\dfrac{x}{3}$?

(A) $-3 \le x \le 3$

(B) $-\dfrac{1}{3} \le x \le \dfrac{1}{3}$

(C) $-2\pi \le x \le 2\pi$

(D) $-\dfrac{\pi}{8} \le x \le \dfrac{\pi}{8}$

QUESTION SEVEN

What is the maximum value of $P = 6\cos\theta + 4\sin\theta$?

(A) 10

(B) 6

(C) $2\sqrt{13}$

(D) $2\sqrt{5}$

QUESTION EIGHT

A particle moves on a line so that its distance from the origin at time t seconds is x cm and its acceleration is given by $\dfrac{d^2x}{dt^2} = 10 - 2x^3$. If v represents the velocity of the particle, and the particle changes direction 1 cm on the negative side of the origin, which of the following equations is correct?

(A) $v^2 = 20x - x^4$

(B) $v^2 = 20x - x^4 + 21$

(C) $v = 10x - \frac{1}{2}x^4$

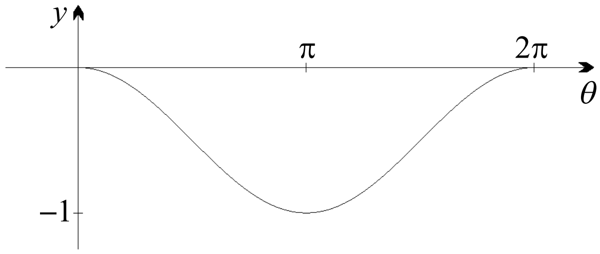
(D) $v = 10x - \frac{1}{2}x^4 + 11\frac{1}{2}$

Examination continues overleaf ...

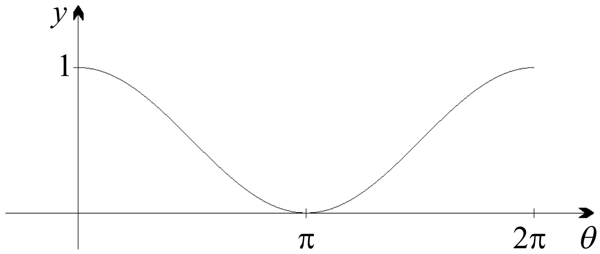
QUESTION NINE

Which of the diagrams below best represents the graph of $y = \cos^2 \frac{1}{2}\theta$?

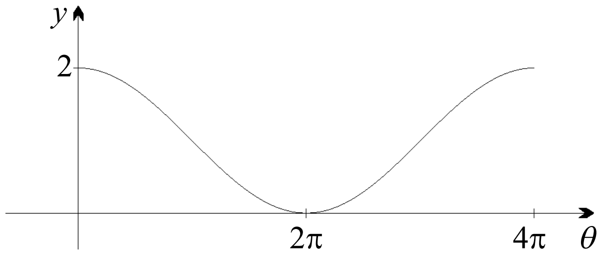
(A)



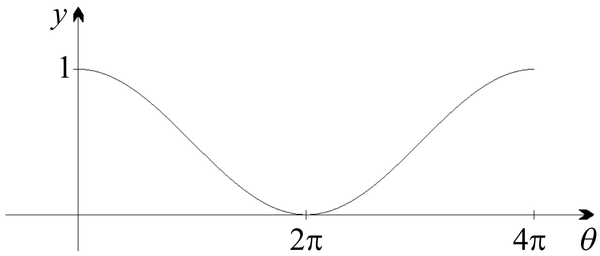
(B)



(C)



(D)



QUESTION TEN

What is the coefficient of z^3 in the expansion of $(1 + z + z^2)^5$?

(A) 10

(B) 20

(C) 30

(D) 40

End of Section I

Examination continues next page ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Find the exact value of $\sin\frac{\pi}{8}\cos\frac{\pi}{8}$. **2**

(b) Evaluate $\sin^{-1}(\sin\frac{4\pi}{3})$. **1**

(c) Show that $\displaystyle\lim_{x\to 0}\frac{\tan\frac{x}{2}}{x} = \tfrac{1}{2}$. **1**

(d) Find the following integrals:

(i) $\displaystyle\int \frac{4x}{16+x^2}\,dx$ **1**

(ii) $\displaystyle\int \frac{3}{9+x^2}\,dx$ **1**

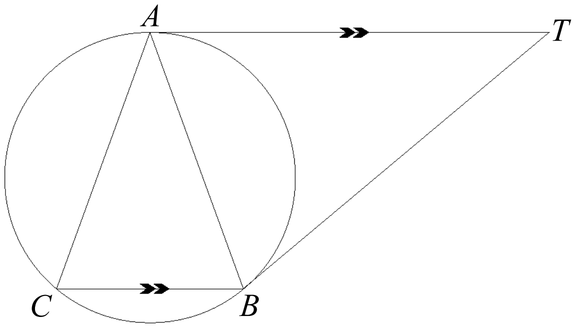
(iii) $\displaystyle\int \frac{-1}{\sqrt{25+x}}\,dx$ **1**

(e) Write down a general solution of the equation $\sin x = -\dfrac{1}{2}$. **1**

(f) If a, b and c are the roots of the equation $3x^3+4x^2-5x-8=0$, find the value of $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$. **2**

(g) By expanding, find the greatest coefficient in the expansion of $(4x+3)^4$. **2**

(h) **3**



Tangents touching a circle at A and B respectively, intersect at T. Point C is on the circle and $AT \parallel CB$. Prove that AB=AC.

Examination continues overleaf ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) An object is put in a freezer to cool. After t minutes, its temperature is T°C. The freezer is at a constant temperature of -8°C. The object's temperature T decreases according to the differential equation $\dfrac{dT}{dt} = -k(T+8)$, where k is a positive constant.

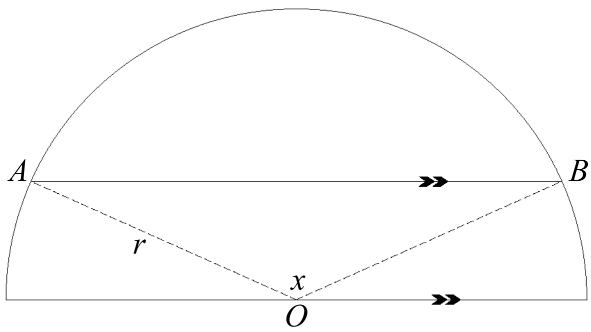
(i) Show that $T = Ae^{-kt} - 8$, where A is a constant, is a solution of the differential equation. **1**

(ii) If the object cools from an initial temperature of 40°C to 30°C in half an hour, find the values of A and k. **2**

(iii) When will the temperature of the object be 0°C? Give your answer correct to the nearest hour. **1**

(iv) Explain what will happen to T eventually. **1**

(b)



The diagram above shows a semi-circle of radius r with centre O. Chord AB is drawn parallel to the base such that it divides the semi-circle into two parts of equal area. Chord AB subtends an angle of x radians at the centre O.

(i) Show that $\sin x = x - \dfrac{\pi}{2}$. **1**

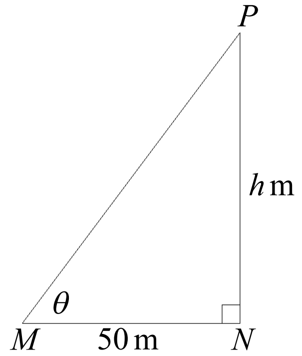
(ii) The equation has a root near $x = 2$. Use one application of Newton's method to find a better approximation for this root, writing your answer correct to three significant figures. **2**

Examination continues next page ...

(c) (i) Use the substitution $u = 3x + 1$ to show that $\displaystyle\int_0^1 \frac{x}{(3x+1)^2}\,dx = \frac{2}{9}\ln 2 - \frac{1}{12}$. **2**

(ii) Hence find the volume of the solid formed when the region bounded by the curve $y = \dfrac{6\sqrt{x}}{3x+1}$, the x-axis and the line $x = 1$ is rotated about the x-axis. Give your answer in exact form. **1**

(d)



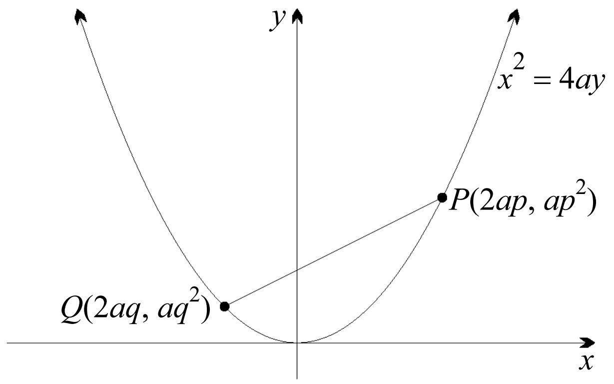
Bowie jumps out of a helicopter and by the time he reaches the position P, h metres above the ground, he is falling at a constant rate of 150 kilometres per hour. Point N is on the ground directly below P and M lies 50 metres from N. The angle of elevation of P from M is θ radians.

(i) Show that $\dfrac{dh}{d\theta} = \dfrac{50}{\cos^2\theta}$. **1**

(ii) Find the rate of decrease of the angle of elevation when Bowie reaches a height of 1200 metres. Give your answer in radians per second. **3**

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola with equation $x^2 = 4ay$.

(i) Find the coordinates of M, the midpoint of PQ. **1**

(ii) Show that the equation of the chord PQ is $y = \frac{1}{2}(p+q)x - apq$. **1**

(iii) If the chord always passes through the point $(0, 2a)$, find the equation of the locus of M. **2**

(b) A particle moves along a straight line and its displacement, x centimetres, from a fixed point O at a given time t seconds is given by $x = 2 + \cos^2 t$.

(i) Show that its acceleration is given by $\ddot{x} = 10 - 4x$. **2**

(ii) Explain why the motion is simple harmonic. **1**

(iii) Find the centre, amplitude and period of the motion. **2**

(c) The polynomial $P(x)$ is given by $P(x) = x^3 - mx^2 + mx - 1$, where m is a constant.

(i) Show that $(x - 1)$ is a factor of $P(x)$. **1**

(ii) Hence find a quadratic factor of $P(x)$. **2**

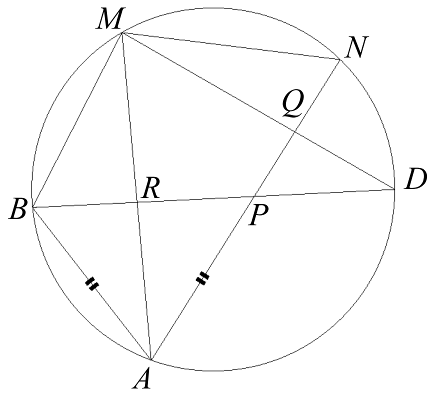
(iii) Hence find the set of values of m for which all the roots of the equation $P(x) = 0$ are real. **2**

(iv) If $m = 3$, the graph of $y = P(x)$ is a transformation of the graph of $y = x^3$. Describe this transformation. **1**

Examination continues next page ...

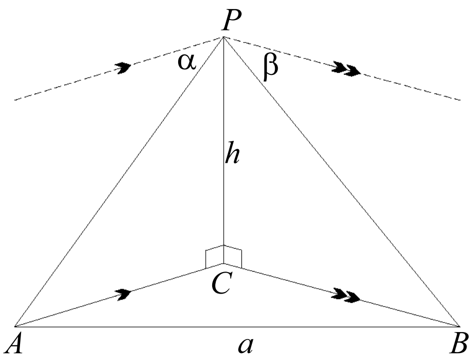
QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) **3**



The diagram above shows a cyclic quadrilateral $ABMN$. Point P lies on AN such that $AB = AP$ and BP produced meets the circle again at D and AM at R. The chord MD intersects AN at Q.
Copy the diagram and show that $QPRM$ is a cyclic quadrilateral.

(b)



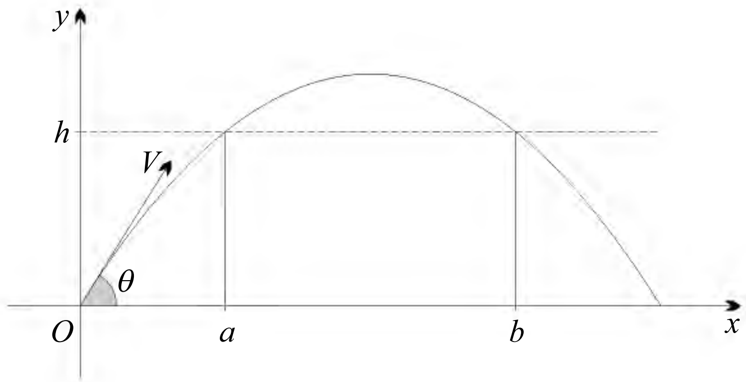
The diagram above shows two points A and B on level ground. B is a metres due east of A. A tower, of height h metres, is also on the same level ground and its bearing is $N\theta E$ and $N\phi W$ from A and B respectively. From the top of the tower P, the angle of depression of A is α and of B is β.

(i) Prove that $h\sin(\theta + \phi) = a\cos\phi\tan\alpha$. **2**

(ii) Prove that $h^2(\cot^2\alpha - \cot^2\beta) - 2ha\cot\alpha\sin\theta + a^2 = 0$. **2**

(c) If $f^{(n)}(x)$ denotes the nth derivative of $f(x) = \dfrac{1}{x}$, prove by mathematical induction that $f^{(n)}(x) = \dfrac{(-1)^n n!}{x^{n+1}}$ for all positive integers n. **3**

(d)



A particle is fired from O with initial velocity V m/s at an angle θ to the horizontal. The particle just clears two thin vertical towers of height h metres at horizontal distances of a metres and b metres from O.

The equations of motion of the particle are $x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$. (Do NOT prove these equations.)

(i) Show that $V^2 = \dfrac{a^2 g(1 + \tan^2\theta)}{2(a\tan\theta - h)}$. **2**

(ii) Hence show that $\tan\theta = \dfrac{h(a+b)}{ab}$. **2**

(iii) Hence show that $\tan\theta = \tan\alpha + \tan\beta$, where α and β are the angles of elevation from O to the tops of the towers. **1**

End of Section II

END OF EXAMINATION

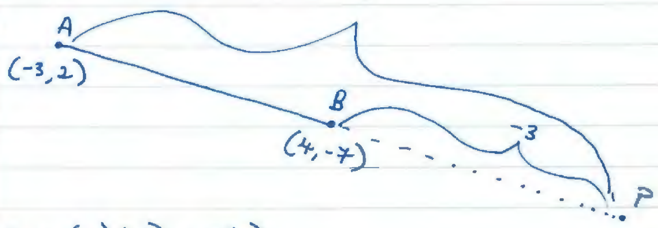
Extension I TRIAL 2017 SOLUTIONS

Multiple Choice:

① $x = 110$ (exterior angle of cyclic quadrilateral ABCD)
$y = 120$ (opposite angles of cyclic quadrilateral ABCD)

choose **D**

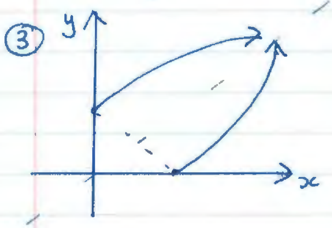
②



$$x = \frac{(-3)(-3) + 5(4)}{5-3}$$
$$= \frac{29}{2}$$
$$= 14\tfrac{1}{2}$$

choose **A**

③



swap x/y
reflect across line $y = x$

choose **D**

④ $y = \sin^{-1}(3x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x)^2}} \times 3$$
$$= \frac{3}{\sqrt{1-9x^2}}$$

choose **C**

⑤ $(x-2)^2 \times \dfrac{x}{x-2} \geqslant 3(x-2)^2$

$$x(x-2) \geqslant 3(x-2)^2$$
$$3(x-2)^2 - x(x-2) \leqslant 0$$
$$(x-2)(3(x-2)-x) \leqslant 0$$
$$(x-2)(2x-6) \leqslant 0$$
$$2(x-2)(x-3) \leqslant 0$$

choose **D**

⑥ $y = 4\sin^{-1}\dfrac{x}{3}$

$$-1 \leqslant x \leqslant 1$$
$$-3 \leqslant x \leqslant 3$$

choose **A**

⑦ $R = \sqrt{6^2 + 4^2}$
$= \sqrt{52}$
$= 2\sqrt{13}$

choose **C**

⑧ $\dfrac{d^2x}{dt^2} = 10 - 2x^3$

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 10 - 2x^3$$
$$\tfrac{1}{2}v^2 = 10x - \tfrac{1}{2}x^4$$
$$v^2 = 20x - x^4 + c$$

at $x = -1$, $v = 0$

$$0 = 20(-1) - (-1)^4 + c$$
$$c = 21$$
$$v^2 = 20x - x^4 + 21$$

choose **B**

(9) using $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$
$= 2\cos^2\frac{\theta}{2} - 1$

$2\cos^2\frac{\theta}{2} = \cos\theta + 1$

$\cos^2\frac{\theta}{2} = \frac{1}{2}\cos\theta + \frac{1}{2}$

so $y = \cos^2\frac{\theta}{2}$ has amplitude $\frac{1}{2}$, range $0 \le y \le 1$, period 2π

choose [B]

(10) $(1+z+z^2)^5 = (1+z(1+z))^5$

now $(1+z(1+z))^5 = {}^5C_0 + {}^5C_1 z(1+z) + {}^5C_2 z^2(1+z)^2 + {}^5C_3 z^3(1+z)^3 + {}^5C_4 z^4(1+z)^4 + {}^5C_5 z^5(1+z)^5$

terms in z^3 come from ${}^5C_2 z^2(1+z)^2$ and ${}^5C_3 z^3(1+z)^3$
term in $z^3 = ({}^5C_2 \times 2 + {}^5C_3 \times 1)\, z^3$
coefficient $= 10 \times 2 + 10 \times 1$
$= 30$

choose [C]

Section II

(11) (a) $\sin\frac{\pi}{8}\cos\frac{\pi}{8} = \frac{1}{2}\sin\frac{\pi}{4}$ ✓
$= \frac{1}{2\sqrt{2}}$ ✓
$= \frac{\sqrt{2}}{4}$

(b) $\sin^{-1}\left(\sin\frac{4\pi}{3}\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
$= -\frac{\pi}{3}$ ✓

(c) $\lim_{x\to 0} \frac{\tan\frac{x}{2}}{x} = \frac{1}{2}\lim_{x\to 0}\frac{\tan\frac{x}{2}}{\frac{x}{2}}$ ✓
$= \frac{1}{2}$

(d) (i) $\int \frac{4x}{16+x^2}\,dx = 2\int\frac{2x}{16+x^2}\,dx$
$= 2\ln(16+x^2) + C$ ✓

(ii) $\int \frac{3}{9+x^2}\,dx = 3 \times \frac{1}{3}\tan^{-1}\frac{x}{3}$
$= \tan^{-1}\frac{x}{3} + C$ ✓

(iii) $\int \frac{-1}{\sqrt{25+x}}\,dx = -1\int (25+x)^{-\frac{1}{2}}\,dx$
$= -1 \times \frac{(25+x)^{\frac{1}{2}}}{\frac{1}{2}}$
$= -2\sqrt{25+x} + C$ ✓

(e) $\theta = n\pi + (-1)^n \sin^{-1}\left(-\frac{1}{2}\right)$
$= n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$ ✓

(f) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{bc+ac+ab}{abc}$
$= \frac{-\frac{5}{3}}{-\left(-\frac{8}{3}\right)}$ ✓
$= -\frac{5}{8}$ ✓

(g) $(4x+3)^4 = (4x)^4 + {}^4C_1(4x)^3(3) + {}^4C_2(4x)^2(3)^2 + {}^4C_3(4x)(3)^3 + 3^4$ ✓

$= 256x^4 + 768x^3 + 864x^2 + 432x + 81$ ✓

the greatest coefficient is 864 ✓

(h) $\angle TAB = \angle ACB$ (angle in the alternate segment) ✓

$\angle TAB = \angle ABC$ (alternate angles, $AT \parallel CB$) ✓

$AC = AB$ (sides opposite equal angles) ✓

15

(12) (a) (i) $T = Ae^{-kt} - 8$

$\frac{dT}{dt} = -kAe^{-kt}$

$= -k(T+8)$ ✓

(ii) at $t=0$, $T=40$

$40 = A - 8$

$A = 48$ ✓

at $t = 30$, $T = 30$

$30 = 48e^{-30k} - 8$

$\frac{38}{48} = e^{-30k}$

$-30k = \log_e\left(\frac{19}{24}\right)$

$k = -\frac{1}{30}\log_e\left(\frac{19}{24}\right)$ ✓

$= \frac{1}{30}\log_e\left(\frac{24}{19}\right)$

(iii) if $T=0$

$0 = 48e^{-kt} - 8$

$\frac{8}{48} = e^{-kt}$

$-kt = \log_e\left(\frac{1}{6}\right)$

$t = -\frac{1}{k}\log_e\left(\frac{1}{6}\right)$

$= 230.091\ldots$ minutes

$= 3.83\ldots$ h

$\doteqdot 4$h ✓

(iv) as $t \to \infty$

$T \to -8$ ✓

(b) (i) $\frac{1}{2}r^2(x - \sin x) = \frac{1}{2} \times \frac{1}{2}\pi r^2$ } ✓

$x - \sin x = \frac{\pi}{2}$

$\sin x = x - \frac{\pi}{2}$

(ii) let $f(x) = \sin x - x + \frac{\pi}{2}$

$f'(x) = \cos x - 1$ ✓

$x_1 = 2 - \dfrac{\sin 2 - 2 + \frac{\pi}{2}}{\cos 2 - 1}$

$= 2.339\ldots$

$\doteq 2.34$ ✓

(c) (i) $\displaystyle\int_0^1 \frac{x}{(3x+1)^2}\,dx$

let $u = 3x + 1$

$\frac{du}{dx} = 3$

$du = 3\,dx$

$3x = u - 1$

$x = \frac{u-1}{3}$

if $x = 0$, $u = 1$

if $x = 1$, $u = 4$

$= \displaystyle\int_1^4 \frac{\frac{u-1}{3}}{u^2} \times \frac{du}{3}$ | some sensible attempt ✓

$= \frac{1}{9}\displaystyle\int_1^4 \frac{u-1}{u^2}\,du$

$= \frac{1}{9}\displaystyle\int_1^4 \frac{1}{u} - u^{-2}\,du$

$= \frac{1}{9}\left[\ln u + \frac{1}{u}\right]_1^4$

$= \frac{1}{9}\left(\ln 4 + \frac{1}{4} - (\ln 1 + 1)\right)$

$= \frac{1}{9}\left(2\ln 2 - \frac{3}{4}\right)$ } ✓

$= \frac{2}{9}\ln 2 - \frac{1}{12}$

(ii) $V = \pi\displaystyle\int_0^1 \left(\frac{6\sqrt{x}}{3x+1}\right)^2 dx$

$= \pi\displaystyle\int_0^1 \frac{36x}{(3x+1)^2}\,dx$

✓ $= 36\pi \times \left(\frac{2}{9}\ln 2 - \frac{1}{12}\right)$

$= \pi(8\ln 2 - 3)$ cubic units

(d) (i) $\tan\theta = \frac{h}{50}$

$h = 50\tan\theta$

$\frac{dh}{d\theta} = 50\sec^2\theta = \frac{50}{\cos^2\theta}$ } ✓

(ii) $\frac{d\theta}{dt} = \frac{d\theta}{dh} \times \frac{dh}{dt}$

$= \frac{\cos^2\theta}{50} \times \frac{125}{3}$ ✓

$= \frac{\frac{1}{577}}{50} \times \frac{125}{3}$ ✓

$= \frac{5}{3462}$ radians/s ✓

150 km/h

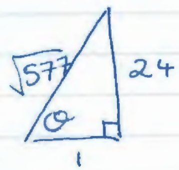
$= \frac{150 \times 1000}{60 \times 60}$

$= \frac{125}{3}$ m/s

if $h = 1200$

$\tan\theta = \frac{1200}{50}$

$= 24$



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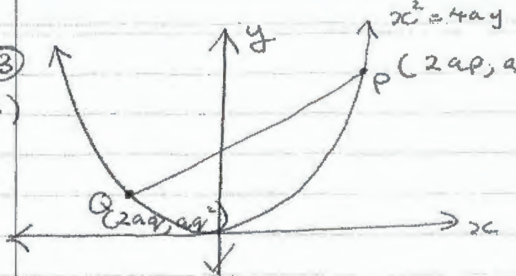
(13)

(a)



(i) $M = \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$= \left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$ ✓

(ii) $m = \frac{ap^2 - aq^2}{2ap - 2aq}$

$= \frac{p^2-q^2}{2(p-q)}$

$= \frac{(p-q)(p+q)}{2(p-q)}$

$= \frac{p+q}{2}, \quad p \neq q$ ✓

equation is $y - ap^2 = \frac{1}{2}(p+q)(x - 2ap)$ ✓

$y = \frac{1}{2}(p+q)x - apq$

(iii) if the chord passes through $(0, 2a)$

$2a = -apq$

$pq = -2$ ✓

so $x^2 = a^2(p+q)^2$

$= a^2(p^2+q^2+2pq)$ ✓

$= a^2\left(\frac{2y}{a} - 4\right)$

$= 2a(y - 2a)$

(b)(i) $x = 2 + \cos^2 t$

$\dot{x} = 2\cos t \times -\sin t$

$= -2\sin t\cos t$

$= -\sin 2t$ ✓

$\ddot{x} = -2\cos 2t$

$= -2(2\cos^2 t - 1)$

$= -4\cos^2 t + 2$

$= -4(x-2) + 2$

$= 10 - 4x$ as required ✓

OR $x = 2 + \cos^2 t$

$= 2 + \frac{1}{2}(\cos 2t + 1)$

$= 2\frac{1}{2} + \frac{1}{2}\cos 2t$ ✓

$\dot{x} = -\sin 2t$

$\ddot{x} = -2\cos 2t$

$= -2\left(x - 2\frac{1}{2}\right) \times 2$ ✓

$= 10 - 4x$ as required

(ii) $\ddot{x} = 10 - 4x$

$= -4\left(x - 2\frac{1}{2}\right)$

which is of the form $\ddot{x} = -n^2(x - x_0)$ ✓

(acceleration is proportional to displacement but in the opposite direction)

OR $x = 2\frac{1}{2} + \frac{1}{2}\cos 2t$

which is just a transformation of $x = \cos t$

so is simple harmonic

(iii) centre : $x = 2\frac{1}{2}$ period $= \frac{2\pi}{2}$

$= \pi$

amplitude $= \frac{1}{2}$

✓ one correct

✓ three correct

(c) $P(x) = x^3 - mx^2 + mx - 1$

(i) $P(1) = 1 - m + m - 1$
$= 0$
so $(x-1)$ is a factor

(ii)

$$\begin{array}{r} x^2 + (1-m)x + 1 \\ x - 1 \,\big)\, x^3 - mx^2 + mx - 1 \\ x^3 - x^2 \\ \hline (1-m)x^2 + mx \quad - 1 \\ (1-m)x^2 - x + mx \\ \hline x - 1 \\ x - 1 \\ \hline 0 \end{array}$$

a solution by inspection is fine

the quadratic factor is $x^2 + (1-m)x + 1$

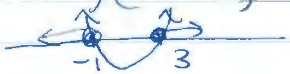
(iii) for real roots we need

$(1-m)^2 - 4(1)(1) \geq 0$

$m^2 - 2m + 1 - 4 \geq 0$

$m^2 - 2m - 3 \geq 0$

$(m-3)(m+1) \geq 0$

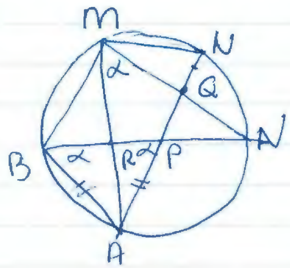


$m \leq -1$ or $m \geq 3$

(iv) If $m = 3$, $P(x) = x^3 - 3x^2 + 3x - 1$
$= (x-1)^3$

This is the graph of $y = x^3$ shifted 1 unit to the right.

(14) (a)



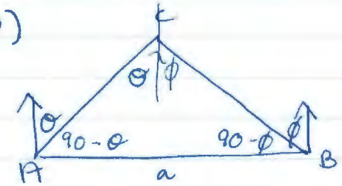
let $\angle ABP = \alpha$

then $\angle PPB = \alpha$ (angles opposite equal sides)

$\angle AMQ = \alpha$ (angles at the circumference on arc AN)

Q, PRM is a cyclic quadrilateral (exterior angle equals opposite interior angles)

(b)



$\angle ACB = \theta + \phi$

(i) in $\triangle ACB$, $\dfrac{\sin(\theta + \phi)}{a} = \dfrac{\sin(90 - \phi)}{AC}$

in $\triangle APC$, $\tan\alpha = \dfrac{h}{AC}$

$AC = \dfrac{h}{\tan\alpha}$

so $\dfrac{\sin(\theta + \phi)}{a} = \dfrac{\sin(90 - \phi)}{\frac{h}{\tan\alpha}}$

$$h\sin(\theta+\phi) = a\sin(90-\phi)\tan\alpha$$

$$h\sin(\theta+\phi) = a\cos\phi\tan\alpha$$

as required

(ii) from $\triangle APC$

$$\cot\alpha = \frac{AC}{h}$$

$$AC^2 = h^2\cot^2\alpha$$

from $\triangle BPC$

$$\cot\beta = \frac{BC}{h}$$

$$BC^2 = h^2\cot^2\beta$$

in $\triangle ABC$

$$\cos(90^\circ-\theta) = \frac{a^2+h^2\cot^2\alpha - h^2\cot^2\beta}{2\times a\times h\cot\alpha}$$

$$\sin\theta = \frac{h^2(\cot^2\alpha-\cot^2\beta)+a^2}{2ha\cot\alpha}$$

$$2ha\cot\alpha\sin\theta = h^2(\cot^2\alpha-\cot^2\beta)+a^2$$

$$h^2(\cot^2\alpha-\cot^2\beta) - 2ha\cot\alpha\sin\theta + a^2 = 0$$

(c) Step 1 : let $n=1$

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

now $f^{(1)}(x) = \frac{(-1)^1 1!}{x^{1+1}}$

$$= -\frac{1}{x^2} \quad \text{as required}$$

the result is true for $n=1$

step 2: suppose k is a positive integer for which the result is true

that is $f^{(k)}(x) = \frac{(-1)^k k!}{x^{k+1}}$

$$= (-1)^k k!\, x^{-(k+1)} \quad *$$

we now prove the result is true for $n=k+1$, that is we prove that

$$f^{(k+1)}(x) = \frac{(-1)^{k+1}(k+1)!}{x^{k+2}}$$

now $f^{(k+1)}(x) = -(k+1)(-1)^k k!\, x^{-(k+1)-1}$ by *

$$= (-1)^{k+1}(k+1)!\, x^{-(k+2)}$$

$$= \frac{(-1)^{k+1}(k+1)!}{x^{k+2}} \quad \text{as required}$$

so by the principle of mathematical induction the result is true for all positive integers n.

(14) (d) $x = Vt\cos\theta$

$y = Vt\sin\theta - \frac{1}{2}gt^2$

(i) at $x = a$, $y = h$

so $a = Vt\cos\theta$

$t = \frac{a}{V\cos\theta}$

and $h = V \times \frac{a}{V\cos\theta}\sin\theta - \frac{1}{2}g\left(\frac{a}{V\cos\theta}\right)^2$ ✓

$h = a\tan\theta - \frac{\frac{1}{2}ga^2}{V^2\cos^2\theta}$

$\frac{ga^2\sec^2\theta}{2V^2} = a\tan\theta - h$

$2V^2 = \frac{ga^2\sec^2\theta}{a\tan\theta - h}$

$V^2 = \frac{ga^2\sec^2\theta}{2(a\tan\theta - h)}$ ✓

$= \frac{ga^2(1+\tan^2\theta)}{2(a\tan\theta - h)}$ (i)

(ii) Similarily, $V^2 = \frac{gb^2(1+\tan^2\theta)}{2(b\tan\theta - h)}$ (ii)

equating (i) + (ii) ✓

$\frac{gb^2(1+\tan^2\theta)}{2(b\tan\theta - h)} = \frac{ga^2(1+\tan^2\theta)}{2(a\tan\theta - h)}$

$b^2(a\tan\theta - h) = a^2(b\tan\theta - h)$

$ab^2\tan\theta - b^2h = ba^2\tan\theta - a^2h$

$ab^2\tan\theta - ba^2\tan\theta = b^2h - a^2h$ ✓

$ab\tan\theta(b-a) = h(b^2-a^2)$

$ab\tan\theta = \frac{h(b-a)(b+a)}{b-a}$

$\tan\theta = \frac{h(b+a)}{ab}$

(iii) $\tan\theta = \frac{hb}{ab} + \frac{ha}{ab}$

$= \frac{h}{a} + \frac{h}{b}$

$= \tan\alpha + \tan\beta$ as required ✓